

# Incidences and pairs of dot products\*

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## Abstract

Let  $\mathbb{F}$  be a field, let  $P \subseteq \mathbb{F}^d$  be a finite set of points, and let  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ . We study the quantity

$$|\Pi_{\alpha, \beta}| = \{(p, q, r) \in P \times P \times P \mid p \cdot q = \alpha, p \cdot r = \beta\}.$$

We observe a connection between the question of placing an upper bound on  $|\Pi_{\alpha, \beta}|$  and a well-studied question on the number of incidences between points and hyperplanes, and use this connection to prove new and strengthened upper bounds on  $|\Pi_{\alpha, \beta}|$  in a variety of settings.

## 1 Introduction

Let  $\mathbb{F}$  be a field, let  $P \subseteq \mathbb{F}^d$  be a finite set of points, and let  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ . Denote

$$\Pi_{\alpha, \beta} = \Pi_{\alpha, \beta}(P) = \{(p, q, r) \in P \times P \times P \mid p \cdot q = \alpha, p \cdot r = \beta\}.$$

The quantity  $\max_{P, \alpha, \beta: |P|=n} (|\Pi_{\alpha, \beta}|)$  was first investigated by Barker and Senger [1], who gave upper bounds on  $|\Pi_{\alpha, \beta}|$  in terms of  $|P|$  for  $P \subset \mathbb{R}^2$ . The case that  $P$  is a sufficiently large subset of a vector space over a finite field, or of a module over the set of integers modulo the power of a prime, was investigated by Covert and Senger [4].

We observe that known upper bounds on the maximum number of incidences between a set of hyperplanes and points imply upper bounds on  $|\Pi_{\alpha, \beta}|$ . We use this approach to obtain new upper bounds on the size of  $|\Pi_{\alpha, \beta}|$  under various restrictions on  $\mathbb{F}$  and  $P$ . In the case  $P \subset \mathbb{R}^2$ , the new bounds strengthen and generalize the results of Barker and Senger.

Our first result shows that  $|\Pi_{\alpha, \beta}| \leq O(n^2)$ , and gives stronger bounds when no line contains too many points of  $P$ .

**Theorem 1.** *Let  $P \subseteq \mathbb{F}^2$  be a set of  $n$  points such that no  $s$  points of  $P$  are collinear, and let  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ . Then,*

$$|\Pi_{\alpha, \beta}| < \min(2s^2n, 4n^2).$$

It is possible that  $|\Pi_{\alpha, \beta}| \geq \Omega(n^2)$ . For example, consider the set of points  $P = p \cup P' \subset \mathbb{R}^2$ , where  $p$  has the coordinates  $(1, 1)$  and  $P'$  is contained in the line  $x + y = 1$ . Then, if  $q, r \in P'$ , we have  $(p, q, r) \in \Pi_{1, 1}$ . Hence,  $|\Pi_{1, 1}| \geq |P'|^2$ .

Barker and Senger [1] showed that  $|\Pi_{\alpha, \beta}| \leq O(n^2)$  when  $P \subset \mathbb{R}^2$ , and Covert and Senger [4] showed that  $|\Pi_{\alpha, \beta}| \leq O(n^2)$  when  $P \subset \mathbb{F}_q^2$ . The observation that we can obtain a better bound if  $P$  contains no large collinear set is new.

In the case that  $P \subset \mathbb{R}^2$  or  $P \subset \mathbb{C}^2$ , we can use the Szemerédi-Trotter theorem [12, 13, 14] to get an improvement to the conclusion of Theorem 1 for  $s \geq \Omega(n^{1/3})$ .

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**Theorem 2.** *Let  $P$  be a set of  $n$  points in  $\mathbb{F}^2$ , for either  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{C}$ . Suppose that no  $s$  points of  $P$  are collinear, and let  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ . Then,*

$$|\Pi_{\alpha, \beta}| \leq O(n^{5/3} + sn)$$

In Section 2, we describe a very simple construction that achieves  $|\Pi_{\alpha, \beta}| = \Omega(sn)$  for any  $s, n$ . This shows that Theorem 2 is tight up to constant factors when  $s \geq n^{2/3}$ . It is an open problem to determine the asymptotically least possible upper bound on  $|\Pi_{\alpha, \beta}|$  for  $s \leq n^{2/3}$ , under the hypotheses of Theorem 2.

It's worth noting that the conclusion of Theorem 2 does not hold for subsets of  $\mathbb{F}_q^2$ . Covert and Senger [4] showed that, for  $P \subseteq \mathbb{F}_q^2$  with  $n = |P| \geq \Omega_\epsilon(q^{3/2+\epsilon})$  for some  $\epsilon > 0$ <sup>1</sup>, we have  $|\Pi_{\alpha, \beta}| = (1 - o(1))(n^3 q^{-2})$ . Hence, in this regime, we have  $|\Pi_{\alpha, \beta}| > \omega(nq^{3/2}) > \omega(n^{5/3}) > \min(2q^2 n, O(n^{5/3} + qn)) = O(n^{5/3})$ .

As a corollary of Theorem 2, we obtain the following improvement to a result of Barker and Senger.

**Corollary 3.** *Let  $P \subset [0, 1]^2$  with  $|P| = n$ , such that the distance between each pair of points in  $P$  is at least  $\epsilon$ . Then,  $|\Pi_{\alpha, \beta}| \leq O(n^{5/3} + n\epsilon^{-1})$ .*

*Proof.* We can assume that  $\epsilon \leq O(n^{1/2})$ , since otherwise  $n$  points cannot be placed in  $[0, 1]^2$  such that the distance between each pair of points is at least  $\epsilon$ . Since each pair of points is at distance at least  $\epsilon$ , and since the maximum distance between any pair of points is at most  $\sqrt{2}$ , there are at most  $O(\epsilon^{-1})$  points on any line. An application of Theorem 2 completes the proof.  $\square$

Under the hypotheses of Corollary 3, Barker and Senger showed that  $|\Pi_{\alpha, \beta}| \leq O(n^{4/3} \epsilon^{-1} \log(\epsilon^{-1}))$ . In Section 2, we give a construction showing that Corollary 3 is tight up to constant factors when  $\epsilon \leq O(n^{1/3})$ .

When  $P \subset \mathbb{F}_q^2$  for  $p$  prime, and  $|P|$  is not too large, we obtain a slight improvement to Theorem 1 when  $s$  is close to  $n^{1/2}$  by using an incidence bound first proved by Bourgain, Katz, and Tao [3, 7, 10].

**Theorem 4.** *Let  $P$  be a set of  $n < p$  points in  $\mathbb{F}_p^2$ , for a prime  $p$ , such that no  $s$  points of  $P$  are collinear. Then, there exists a constant  $\epsilon > 0$  such that*

$$|\Pi_{\alpha, \beta}| \leq O(sn^{3/2-\epsilon}).$$

For  $P \subset \mathbb{F}^d$  with  $d > 2$ , there is no upper bound of the form  $|\Pi_{\alpha, \beta}| \leq o(n^3)$  that holds for an arbitrary set of  $n$  points. For example, let  $A \subset \mathbb{R}$  with  $|A| = n/2$ , and let  $P$  be all points with coordinates  $(a, 0, \beta)$  or  $(0, a, 1)$ , for  $a \in A$  and  $\beta \in \mathbb{R} \setminus \{0\}$ . If  $p \in P$  has the form  $(a, 0, \beta)$  and  $q, r \in P$  have the form  $(0, a, 1)$ , then  $(p, q, r) \in \Pi_{\beta, \beta}$ . Since we have  $n/2$  choices for  $p$ , and  $(n/2)^2$  choices for  $(q, r)$ , we have that  $|\Pi_{\beta, \beta}| \geq \Omega(n^3)$ .

It is possible to obtain a nontrivial upper bound on  $|\Pi_{\alpha, \beta}|$  for  $P \subset \mathbb{F}^d$  by restricting the maximum possible number of points on a hyperplane. We obtain a more refined bound by further restricting the number of points on a  $(d-2)$ -plane<sup>2</sup>.

**Theorem 5.** *Let  $P$  be a set of  $n$  points in  $\mathbb{F}^d$ , such that no  $s$  points of  $P$  are contained in any single hyperplane, and such that no  $t$  points of  $P$  are contained in any single  $(d-2)$ -plane. Let  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ . Then,*

$$|\Pi_{\alpha, \beta}| \leq \min(2s^2 n, O(tn^2)).$$

Up to constant factors, Theorem 5 is a generalization of Theorem 1; since the points of  $P$  are distinct,  $t = 1$  in  $\mathbb{F}^2$ .

Rudnev [11] proved an upper bound on the number of incidences between points and planes in  $\mathbb{F}^3$  that holds for an arbitrary field  $\mathbb{F}$  with characteristic other than 2. In the case of positive characteristic, application of Rudnev's bound requires that there are not too many points. Applying Rudnev's bound in the framework of this paper gives

<sup>1</sup>The subscript in the notation  $O_\epsilon$  indicates that the constants hidden in the  $O$ -notation depend on  $\epsilon$ .

<sup>2</sup>We refer to a  $k$  dimensional affine subspace as a  $k$ -plane. A hyperplane in  $\mathbb{F}^d$  is a  $(d-1)$ -plane.

**Theorem 6.** *Let  $P$  be a set of  $n$  points in  $\mathbb{F}^3$ , such that no  $s$  points of  $P$  are coplanar, and no  $t$  points of  $P$  are collinear. If  $\mathbb{F}^3$  has positive characteristic  $p$ , then  $p \neq 2$  and  $n = O(p^2)$ . Then,*

$$|\Pi_{\alpha,\beta}| \leq O(n^2 \log(sn^{-1/2}) + stn).$$

Theorem 6 gives a stronger conclusion than Theorem 5 for  $s \geq \Omega(n^{1/2})$ .

The polynomial partitioning technique of Guth and Katz [6] has recently led to a number of higher-dimensional incidence bounds in  $\mathbb{R}^d$ . Applying one such bound [9], we find

**Theorem 7.** *Let  $P$  be a set of  $n$  points in  $\mathbb{R}^d$ . Suppose no more than  $s$  points lie on any  $(d-1)$ -plane, and no more than  $t$  points lie on any  $(d-2)$ -plane. Then, for any  $\epsilon > 0$ ,*

$$|\Pi_{\alpha,\beta}| \leq O_{\epsilon,d}(nt^2 + n^{(4d-3)/(2d-1)+\epsilon} t^{(2d-2)/(2d-1)+\epsilon} + sn).$$

Theorem 7 falls short of being a generalization of Theorem 2 in two respects. First, it is only proved in real space, not complex space. Second, it is weaker by an arbitrarily small polynomial factor  $(nt)^\epsilon$ . Both of these limitations are inherited from the incidence bound used, and could conceivably be removed by future developments in the science of proving incidence bounds.

## 2 Constructions

In this section, we describe two infinite families of sets of points in  $\mathbb{R}^2$ . The first family is relatively simple, and shows that a set of  $n$  points such that no line contains  $s$  of the points can have  $|\Pi_{\alpha,\beta}| \geq ns$ . This shows that Theorem 2 is tight for  $s \geq n^{2/3}$ .

The second construction shows that that Corollary 3 is tight. This also implies that Theorem 2 is tight for  $s \geq n^{2/3}$ , but says nothing for  $s < n^{1/2}$ . The second construction is also slightly more complicated than the first, and results in fewer pairs of dot products by a constant factor.

In both this section and Section 3, we will need the following observation.

**Lemma 8.** *Let  $p, q \in \mathbb{F}^d$  and  $\alpha \in \mathbb{F} \setminus \{0\}$ . The set of points  $\ell_p = \{r \mid r \cdot p = \alpha\}$  is a hyperplane. If  $p, q$  are distinct, then  $\ell_p, \ell_q$  are distinct.*

*Proof.* Write  $p = (p_1, \dots, p_d), r = (r_1, \dots, r_d), q = (q_1, \dots, q_d)$ . The set of points  $\ell_p$  satisfies the linear equation

$$p_1 r_1 + \dots + p_d r_d = \alpha,$$

and so is a hyperplane. Suppose  $\ell_p = \ell_q$ . Then there is some  $\beta$  such that  $\beta(p_1, \dots, p_d, \alpha) = (q_1, \dots, q_d, \alpha)$ . Since  $\alpha \neq 0$ , we must have  $\beta = 1$ , so  $p = q$ .  $\square$

### 2.1 Simple construction for Theorem 2

For given positive integers  $n, s$  such that  $n/s$  is an integer, we construct a set  $P \subset \mathbb{R}^2$  of  $n$  points, no  $s$  collinear, such that  $|\Pi_{1,1}| \geq n(s-1)^2/s$ .

Let  $Q = \{q_1, q_2, \dots, q_{n/s}\}$  be a set of points such that no three are on a line, and such that  $q_i \neq (0,0)$  for all  $i$ . For  $i \in [1, n/s]$ , let  $\ell_i = \{p \mid p \cdot q_i = 1\}$ . Let  $R_i$  be a set of  $s-1$  points such that, if  $r \in R_i$ , then  $r \in \ell_i$ , and no  $s$  points of  $P = Q \cup R_1 \cup R_2 \cup \dots \cup R_{n/s}$  are collinear. Note that  $|P| = n$ . For each pair  $(r_{ij}, r_{ik}) \in R_i$ , we have  $(q_i, r_{ij}, r_{ik}) \in \Pi_{1,1}$ . Since there are  $n/s$  choices for  $q_i$  and  $(s-1)^2$  choices for  $(r_{ij}, r_{ik})$ , we have that  $|\Pi_{1,1}| \geq n(s-1)^2/s$ .

### 2.2 Construction for Corollary 3

For given  $n$  and  $\epsilon < n^{-1/2}/3$  satisfying certain divisibility conditions, we construct a set  $P \subset [0,1]^2$  of  $n + 3n\epsilon = (1 + o(1))n$  points such that the distance between each pair of points in  $P$  is at least  $\epsilon$ , such that  $|\Pi_{1/2,1/2}| \geq \Omega(n\epsilon^{-1})$ .

Let  $L = \{\ell_1, \ell_2, \dots, \ell_{3n\epsilon}\}$  be a set of lines such that  $\ell_j$  contains the points  $(0, 1)$  and  $(1, 1 - 3\epsilon j)$ . Note that, since  $\epsilon < n^{-1/2}/3$ , each line of  $L$  has positive  $y$ -coordinate for all  $x \in [0, 1]$ . Let  $Q_i$  for  $i \in [1, 3n\epsilon]$  be a set of  $\epsilon^{-1}/3$  points such that each point of  $Q_i$  is incident to  $\ell_i$ , has  $x$ -coordinate in the interval  $[2/3, 1]$ , and the difference between the  $x$ -coordinate of each pair of points in  $Q_i$  is at least  $\epsilon$ . Note that the distance between  $\ell_i$  and  $\ell_{i+1}$  at  $x = 2/3$  is  $(2/3)3\epsilon > \epsilon$ , so the distance between points in  $Q_i$  and  $Q_{i+1}$  is at least  $\epsilon$ . Let  $R = \{r_1, r_2, \dots, r_{3n\epsilon}\}$  be the set of points such that the coordinates of  $r_j$  are  $(3\epsilon j/2, 1/2)$ . Note that the distance between each pair of points in  $R$  is at least  $3\epsilon/2$ , and all points of  $R$  have  $x$ -coordinate at most  $9n\epsilon^2/2 < 1/2$ . Hence,  $P = Q_1 \cup Q_2 \cup \dots \cup Q_{3n\epsilon} \cup R$  is a set of  $(1 + o(1))n$  points such that the distance between each pair of points in  $P$  is at least  $\epsilon$ .

Let  $q \in Q_j$  with coordinates  $(\lambda, 1 - 3\lambda\epsilon j)$  for some  $\lambda \in [2/3, 1]$ . Then  $r_j \cdot q = \lambda 3\epsilon j/2 + (1 - \lambda 3\epsilon j)/2 = 1/2$ . Hence, for  $q_{j1}, q_{j2} \in Q_i$ , we have  $(r_j, q_{j1}, q_{j2}) \in \Pi_{1/2, 1/2}$ . Since there are  $3n\epsilon$  choices for  $r_j$  and  $\epsilon^{-2}/9$  choices for  $(q_{j1}, q_{j2})$ , we have  $|\Pi_{1/2, 1/2}| \geq n\epsilon^{-1}/3$ .

### 3 Proofs

In this section, we prove the main theorems stated in Section 1.

The proofs all have the same basic outline. First, we use a unified reduction from the question studied here to an incidence problem; this reduction is in Section 3.1. Section 3.1 also introduces notation that is used in the subsequent proofs. Then, we apply known incidence bounds to obtain the concrete results listed in Section 1. These six proofs are organized into two sections; Section 3.2 includes proofs of those bounds that are proved for point sets in a plane, and Section 3.3 has the proofs for bounds in higher dimensions.

#### 3.1 From Pairs of Dot Products to Incidences

Suppose we are given a finite point set  $P \subset \mathbb{F}^d$  such that no hyperplane contains  $s$  points of  $P$ , and constants  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ .

For any hyperplane  $h$ , denote

$$wt(h) = |h \cap P|.$$

For any point  $p \in P$  and constant  $c \in \mathbb{F}$ , denote

$$\begin{aligned} h_c(p) &= \{x \in \mathbb{F}^d \mid p \cdot x = c\}, \\ \pi(p) &= \{(q, r) \in P \times P \mid p \cdot q = \alpha, p \cdot r = \beta\}. \end{aligned}$$

Note that  $|\Pi_{\alpha, \beta}| = \sum_{p \in P} |\pi(p)|$ , and  $|\pi(p)| = wt(h_\alpha(p)) \cdot wt(h_\beta(p)) < wt(h_\alpha(p))^2 + wt(h_\beta(p))^2$ . Hence,

$$|\Pi_{\alpha, \beta}| < \sum_{p \in P} wt(h_\alpha(p))^2 + \sum_{p \in P} wt(h_\beta(p))^2.$$

Let  $\gamma = \arg \max_{\gamma \in \{\alpha, \beta\}} \sum_{p \in P} wt(h_\gamma(p))^2$ ; we have

$$|\Pi_{\alpha, \beta}| < 2 \sum_{p \in P} wt(h_\gamma(p))^2.$$

Let

$$H = \{h_\gamma(p) \mid p \in P\}.$$

Since  $\gamma \neq 0$ , if  $p \neq p'$  then  $h_\gamma(p) \neq h_\gamma(p')$ , so  $|H| = n$ .

Denote

$$\begin{aligned} f_k &= |\{h \in H \mid wt(h) \geq k\}|, \\ f_{=k} &= |\{h \in H \mid wt(h) = k\}|. \end{aligned}$$

Collecting hyperplanes of equal weight, we have

$$\sum_{p \in P} wt(h_\gamma(p))^2 = \sum_{k < s} f_{=k} k^2.$$

We have now established

**Lemma 9.** *Let  $P \subset \mathbb{F}^d$  be a finite set of points such that no hyperplane contains more than  $s$  points of  $P$ , and let  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ . Then,*

$$|\Pi_{\alpha, \beta}| < 2 \sum_{k < s} f_{=k} k^2.$$

Since  $|H| = n$ , we have as an immediate corollary to Lemma 9

**Corollary 10.** *Under the hypotheses of Lemma 9, we have*

$$|\Pi_{\alpha, \beta}| < 2s^2 n.$$

Let  $g_k$  be a monotonically decreasing function of  $k$ . We claim that, if  $f_k \leq g_k$  for all  $k$ , then  $\sum_{k < s} k^2 f_{=k} = \sum_{k < s} k^2 (f_k - f_{k+1}) \leq \sum_{k < s} k^2 (g_k - g_{k+1})$ . The proof is by induction on the minimum index  $j$  such that  $f_i = g_i$  for all  $i > j$ . In the base case,  $f_k = g_k$  for all  $k$ . If  $f_j \neq g_j$ , then  $f_j < g_j$ . The function  $f'$  such that  $f'_k = f_k$  for  $k \neq j$  and  $f'_j = g_j$  has the property that  $\sum_{k \leq s} k^2 (f_k - f_{k+1}) \leq \sum_{k \leq s} k^2 (f'_k - f'_{k+1})$ , and by induction  $\sum_{k \leq s} k^2 (f'_k - f'_{k+1}) \leq \sum_{k \leq s} k^2 (g_k - g_{k+1})$ .

Hence, we have

**Lemma 11.** *Let  $P \subset \mathbb{F}^d$  be a finite set of points such that no hyperplane contains more than  $s$  points of  $P$ , and let  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ . Let  $g_k$  be a monotonically decreasing function of  $k$  such that  $g_k \geq f_k$  for all  $k$ . Then*

$$|\Pi_{\alpha, \beta}| \leq 2 \sum_{k \leq s} k^2 (g_k - g_{k+1}).$$

In the proofs, we will use the following specific function  $g_k$  with Lemma 11, which is related to the maximum number of hyperplanes that can contain at least  $k$  of a set of  $n$  points.

Let  $\mathcal{H}$  be the set of all hyperplanes in  $\mathbb{F}^d$ , and denote

$$g'_k = |\{h \in \mathcal{H} : wt(h) \geq k\}|,$$

$$g_k = \begin{cases} g'_k, & \text{if } g'_k \leq n, \\ n, & \text{otherwise.} \end{cases}$$

Since each hyperplane that contributes to  $f_k$  also contributes to  $g_k$ , we have that  $g_k \geq f_k$ . In addition,  $g_k$  is monotonically decreasing, and so satisfies the hypotheses of Lemma 11.

We denote

$$g_{=k} = g_k - g_{k+1};$$

in particular, this implies that  $\sum_k g_{=k} \leq n$ .

In the following proofs, we will often derive a bound on  $g_k$  from some known bound on the quantity

$$I(P, H) = |\{(p, h) \in P \times H \mid p \in h\}|,$$

in which  $P$  and  $H$  may be taken to be arbitrary sets of points and hyperplanes, respectively.

### 3.2 On a plane

In this section, we prove Theorems 1, 2, and 4. These theorems are all for planar point sets, and are united by a common hypothesis (no  $s$  points on any line).

### 3.2.1 Theorem 1

The proof for arbitrary field  $\mathbb{F}$  uses only the facts that each distinct pair of points lies on one line, each distinct pair of lines intersects in at most one point, and  $g_k \leq n$  for all  $k$ .

*Proof of Theorem 1.* There are  $k^2$  ordered pairs of (not necessarily distinct) points of  $P$  on a line containing  $k$  points of  $P$ , and  $n^2$  such pairs in total. Each distinct pair of points appears on one line, and each line crosses at a single point, so

$$\sum_{k \geq 2} k^2 g_k \leq n^2 + \binom{g_2}{2} < 2n^2.$$

Combined with Lemma 9 and Corollary 10, this completes the proof.  $\square$

### 3.2.2 Theorem 2

For  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ , we can use the Szemerédi-Trotter theorem, proved for  $\mathbb{R}$  by Szemerédi and Trotter [12]. Since the same bound was proved for  $\mathbb{C}$  by Tóth [13], and later by completely different methods by Zahl [14], we have a unified proof for  $\mathbb{F} = \mathbb{R}$  and  $\mathbb{C}$ .

**Lemma 12** (Szemerédi-Trotter). *For  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{C}$ ,*

$$g_k \leq O(n^2/k^3 + n/k).$$

*Proof of Theorem 2.* Note that, since  $\sum_k g_k \leq n$ ,

$$\sum_{k \leq n^{1/3}} k^2 n \leq n^{5/3}.$$

Combining this with Lemmas 11 and 12,

$$\begin{aligned} |\Pi_{\alpha, \beta}| &\leq n^{5/3} + O\left(\sum_{n^{1/3} \leq k \leq s} (n^2/k^2 + n)\right) \\ &\leq n^{5/3} + O(n^{5/3} + ns). \end{aligned}$$

$\square$

### 3.2.3 Theorem 4

In the case when  $\mathbb{F}$  is a finite field with prime order, we can use an incidence theorem that was first proved by Bourgain, Katz, and Tao [3] to obtain a slight improvement over Theorem 1 when  $|P|$  is not too large.

**Lemma 13.** *Let  $P$  be a set of points and  $L$  be a set of lines in  $\mathbb{F}_p^2$  for prime  $p$ , with  $|P|, |L| \leq N < p$ . Then there is a constant  $\epsilon > 0$  such that  $I(P, L) \leq O(N^{3/2-\epsilon})$ .*

The value of  $\epsilon$  in Lemma 13 was improved by Jones [7], and recent improvements to the sum-product theorem in  $\mathbb{F}_p^2$  by Roche-Newton, Rudnev, and Shkredov [10] give further improvements to  $\epsilon$ .

*Proof of theorem 4.* Lemma 13 implies that

$$kg_k \leq n^{3/2-\epsilon}.$$

Hence, by Lemma 11,

$$|\Pi_{\alpha, \beta}| \leq 2 \sum_{k \leq s} O(n^{3/2-\epsilon}) \leq O(sn^{3/2-\epsilon}).$$

$\square$

### 3.3 Higher dimensions

In this section, we prove Theorems 5, 6, and 7. These theorems are for sets of points in some higher dimensional space, and are united by the hypotheses that no  $s$  points are contained in a hyperplane and no  $t$  points are contained in a  $(d-2)$ -plane.

Given a set  $H$  of hyperplanes and a set  $P$  of points, we define the incidence graph  $G(H, P)$  to be the bipartite graph with left vertices corresponding to the hyperplanes of  $H$ , right vertices corresponding to the points of  $P$ , and  $(h, p) \in E(G)$  if and only if  $p \in h$ . We denote the complete bipartite graph with  $s$  left and  $t$  right vertices as  $K_{s,t}$ .

**Lemma 14.** *Suppose  $H$  is a set of hyperplanes in  $\mathbb{F}^d$ , and  $P$  is a set of points, such that no  $t$  points are contained in any single  $(d-2)$ -plane. Then the incidence graph  $G(H, P)$  does not include  $K_{2,t}$  as a subgraph.*

*Proof.* Since the hyperplanes of  $H$  are distinct, the intersection of any two hyperplanes of  $H$  is a  $(d-2)$ -plane, which does not contain  $t$  points by hypothesis.  $\square$

#### 3.3.1 Theorem 5

The classic bound of Kővári, Sós, and Turán [8] gives an upper bound on the number of edges in a  $K_{2,t}$ -free graph.

**Lemma 15** (Kővári-Sós-Turán). *Let  $G$  be a  $K_{2,t}$ -free bipartite graph with  $m$  left vertices and  $n$  right vertices. Then the number of edges of  $G$  is at most  $O(t^{1/2}mn^{1/2} + n)$ .*

From this, we can derive an upper bound on  $|\Pi_{\alpha,\beta}|$  for a set of points in  $\mathbb{F}^d$ , for an arbitrary field  $\mathbb{F}$ .

*Proof of Theorem 5.* Since no  $t$  points of  $P$  lie in any  $(d-2)$ -plane, Lemma 14 implies that the incidence graph of  $P$  with an arbitrary set of hyperplanes is  $K_{2,t}$ -free. Hence, by Lemma 15,

$$kg_k \leq O(t^{1/2}g_k n^{1/2} + n).$$

Hence, either  $k \leq O(t^{1/2}n^{1/2})$ , or  $g_k \leq n/k$ . For  $k \leq O(t^{1/2}n^{1/2})$ , since  $\sum_k g_k \leq n$ , we have

$$\sum_{k \leq t^{1/2}n^{1/2}} k^2 g_k \leq tn^2.$$

When  $s \geq O(t^{1/2}n^{1/2})$ , by Lemma 11 we have

$$|\Pi_{\alpha,\beta}| \leq O(tn^2) + \sum_{k \leq s} O(n) \leq O(tn^2),$$

since  $s \leq n$ . The term  $2s^2n$  in the bound comes from Corollary 10.  $\square$

#### 3.3.2 Theorem 6

Rudnev gave an improvement to Lemma 15 for incidences between points and planes in  $\mathbb{R}^3$ , under the condition that, if  $\mathbb{F}$  has positive characteristic  $p$ , then the number of planes is  $O(p^2)$ .

**Lemma 16.** *Let  $P, H$  be sets of points and planes, of cardinalities respectively  $n$  and  $m$ , in  $\mathbb{F}^3$ . Suppose  $n \geq m$ , and if  $\mathbb{F}$  has positive characteristic  $p$ , then  $p \neq 2$  and  $m = O(p^2)$ . Let  $t$  be the maximum number of collinear planes. Then,*

$$I(P, H) \leq O(n\sqrt{m} + tn).$$

*Proof of Theorem 6.* Since the  $g_k \leq n$ ,

$$\sum_{k < n^{1/2}} k^2 g_{=k} \leq n^2.$$

Lemma 16 gives

$$kg_k \leq O(n\sqrt{g_k} + tn).$$

For  $s \geq n^{1/2}$ ,

$$\begin{aligned} \sum_{k < s} k^2 (g_k - g_{k+1}) &\leq n^2 + \sum_{n^{1/2} < k < s} O(n^2/k + tn), \\ &\leq O(n^2 \log(sn^{-1/2}) + stn). \end{aligned}$$

Combined with Lemma 11, this finishes the proof.  $\square$

### 3.3.3 Theorem 7

For  $\mathbb{F} = \mathbb{R}$ , we can use the following special case of an incidence bound of Lund, Sheffer, and de Zeeuw [9], based on the work of Fox, Pach, Sheffer, Suk, and Zahl [5]. For the special case  $d = 3$ , an earlier result of Basit and Sheffer would be sufficient for our purposes [2].

**Lemma 17.** *Let  $H$  be a set of  $m$  hyperplanes, and  $P$  a set of  $n$  points, both in  $\mathbb{R}^d$ , such that the incidence graph  $G(H, P)$  is  $K_{2,t}$ -free. Then, for any  $\epsilon > 0$ ,*

$$I(H, P) \leq O_{d,\epsilon} \left( m^{2(d-1)/(2d-1)+\epsilon} n^{d/(2d-1)} t^{(d-1)/(2d-1)} + tm + n \right).$$

*Proof of Theorem 7.* By Lemma 17,

$$kg_k \leq O_{d,\epsilon} \left( g_k^{2(d-1)/(2d-1)+\epsilon} n^{d/(2d-1)} t^{(d-1)/(2d-1)} + tg_k + n \right).$$

For  $k \leq O(t)$ , this is trivial.

Let  $\epsilon' = (2d-1)\epsilon/(1-(2d-1)\epsilon)$  — note that  $\epsilon'$  is a function of  $\epsilon$  that increases monotonically for  $\epsilon > 0$ , and that  $\lim_{\epsilon \rightarrow 0} \epsilon' = 0$ .

Let

$$r = \max(t, (nt)^{(1+2\epsilon')(d-1)/(2d-1)}).$$

Since  $g_k \leq n$ ,

$$\sum_{k \leq r} g_{=k} k^2 \leq nr^2.$$

For  $k \geq \Omega(r)$ , we have

$$g_k \leq O_{d,\epsilon} \left( (n^d t^{d-1} / k^{2d-1})^{1+\epsilon'} + n/k \right).$$

Hence,

$$\begin{aligned} \sum_{k < s} k^2 (g_k - g_{k+1}) &\leq O_{d,\epsilon} \left( nr^2 + \sum_{r < k < s} \left( (n^d t^{d-1} k^{-2d+2})^{1+\epsilon'} + n \right) \right), \\ &\leq O_{d,\epsilon} \left( nr^2 + (n^d t^{d-1} r^{-2d+3})^{1+\epsilon'} + ns \right), \\ &\leq O_{d,\epsilon} \left( nt^2 + (n^{(4d-3)/(2d-1)} t^{(2d-2)/(2d-1)})^{1+2\epsilon'} + ns \right). \end{aligned}$$

Applying Lemma 11 completes the proof.  $\square$



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